Lecture 9

Plan: 1) Finish polyhedra
2) Preview applications.

Polyhedra Cont.

Recall: Nonredundant = Facets.

- inequality $a_i^T \leq b_i$ redundant if P unchanged when it's removed.

- $I_= := \{i : a_i^T x = b_i \ \forall x \in P\}$ "equalities"

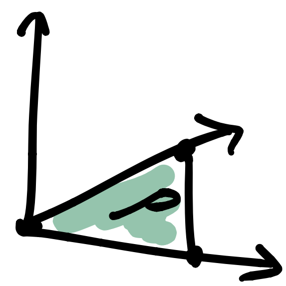
- $\{i : \exists x \in P \ a_i^T x < b_i\}$

□ $I_< := \{ i : \exists x \in P \;\; a_i^T x < b_i \}$.

"real inequalities"

<u>e.g.</u>

$$P = \left\{ x : \begin{array}{l} \left.\begin{array}{r} x_1 + x_2 \le 1 \\ -x_1 \le 0 \\ -x_2 \le 0 \end{array}\right] I_< \\ \left.\begin{array}{r} x_3 \le 0 \\ -x_3 \le 0 \end{array}\right] I_= \end{array}\right.$$

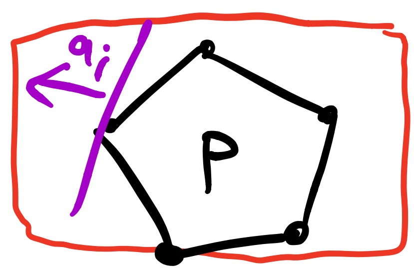


THEN:

$\boxed{\text{Not facet} \Rightarrow \text{redund.}}$

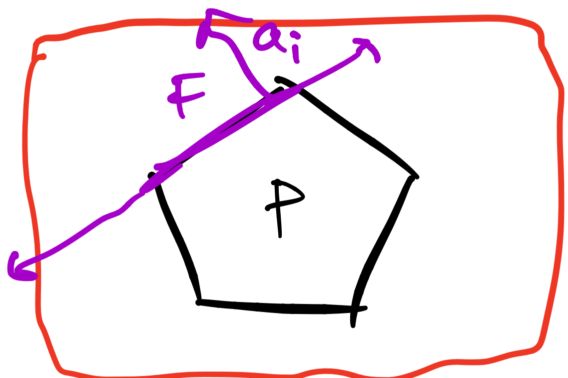
face $a_i^T x = b_i$ for $i \in I_<$ <u>not facet</u>

$\Rightarrow$ $a_i^T x \le b_i$ is <u>redundant</u>.



(need $i \in I_<$, e.g. $x_3 \ge 0$, $x_3 \le 0$ in example neither facets nor redundant.)

Facet ⇒ non redund. F is facet of P, ⇒
$\exists i \in I_<$ s.t. F from $a_i^T x = b_i$.



TAKE-HOME: in minimal description of P, need
- lin-indep set of equalities ($I_=$)
- one inequality per facet ($I_<$).

Proof We only prove ⇒.

- Suppose $a_i^T x \le b_i$ not redundant * want to show corresp. face F_i facet.

- We'll do this by showing

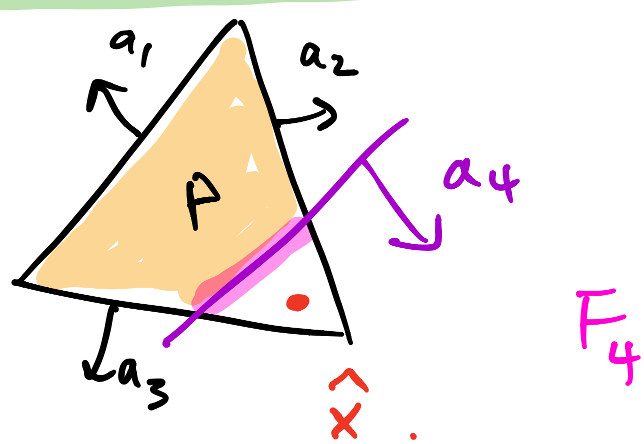
$$\boxed{\begin{array}{l} \dim(F) \geq \dim(P) - 1 \\ \& \ \dim(F) \neq \dim(P). \end{array}}$$

$\boxed{\dim(F) \geq \dim(P) - 1}$:

- $* \Rightarrow$ Is $\hat{x}$ s.t.

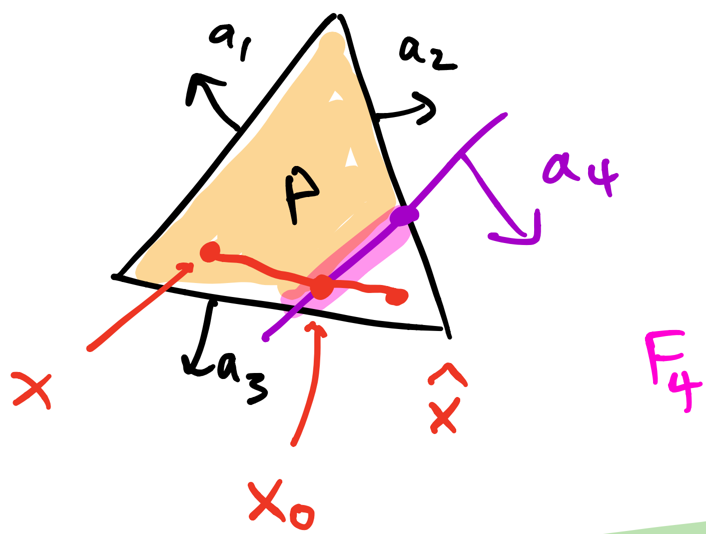
$$\text{but } \boxed{\begin{array}{l} a_i^T \hat{x} > b_i \\ a_j^T \hat{x} \leq b_j \quad \forall j \neq i. \end{array}}$$

e.g. $i = 4$



- Let F_i be face $a_i^T x = b_i$.
- $\forall x \in P$, line segment $x \to x_0$ has unique $x_0 \in F_i$.

e.g. $i=4$



$\Rightarrow$ any point $x \in P$ contained in

$$\mathrm{aff}(F_i, x_0)!$$

- $P \subseteq \mathrm{aff}(F_i, x_0) \Rightarrow \dim(P) \le \dim(F_i) + 1.$

$\boxed{\dim(F) \neq \dim(P):}$

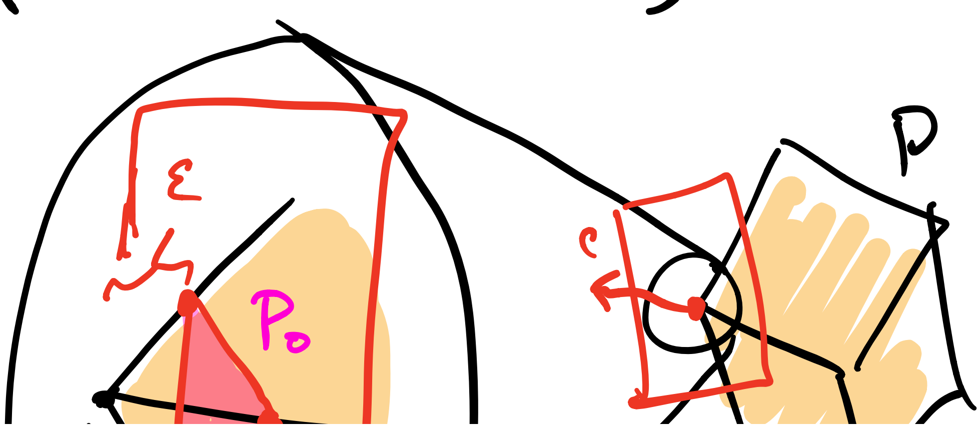
- Recall $i \in I_<$.

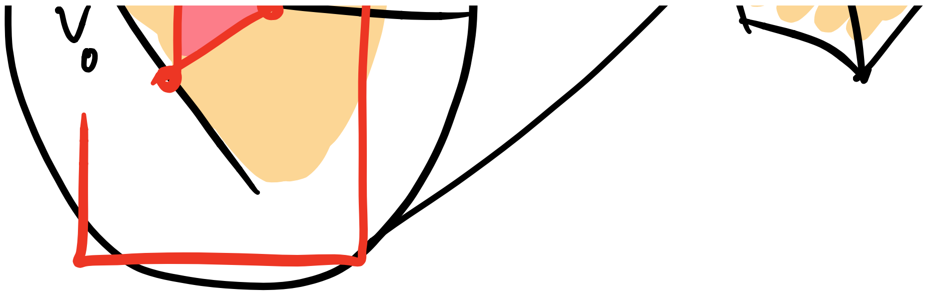
$\Rightarrow$ is point $x_< \in P$ with $a_i^T x < b_i$.

- $x_<$ can't be in $\mathrm{aff}(F_i)$. $\square$

Recall: Near vertex = Cone(polytope)

(N.V.C. Theorem)





Let v_0 vertex of P from valid inequality $c^T x \le m$.

Let ε be such that $\boxed{c^T v' \le m - \varepsilon}$ for all other vertices v'.

Then

$$P_0 = \{ x \in P : c^T x = m - \varepsilon \}$$

is a polytope & is bijection

$\{ P_0\text{'s dim } k \text{ faces} \} \longleftrightarrow$

$\{ P\text{'s dim } k+1 \text{ faces}$

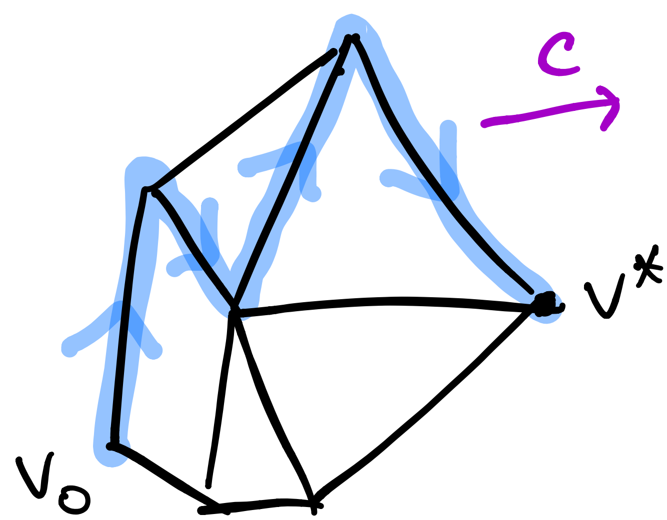
{P's ai…

containing v_0 }

Corollary: Graph connected

Graph of vertices & edges of polyhedron P is always connected.

In particular: if v^* max. of $c^T x$ over P, v_0 vertex, $\exists\ v_0 \to v^*$ path which doesn't decrease objective.

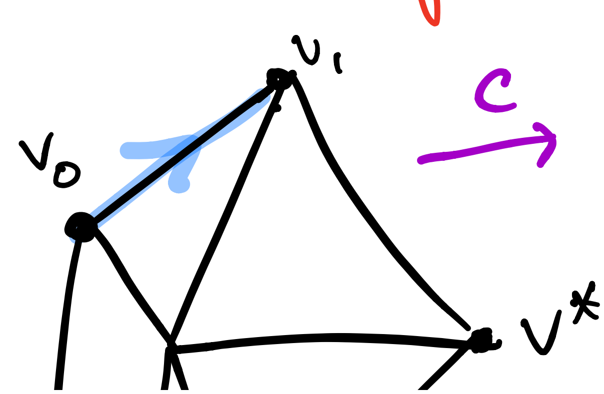


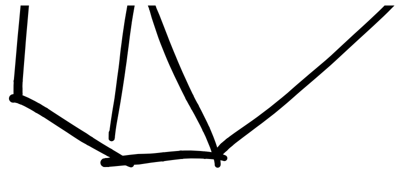
Proof of Corollary:

- Suppose v^* unique max of $c^T x$ over P.

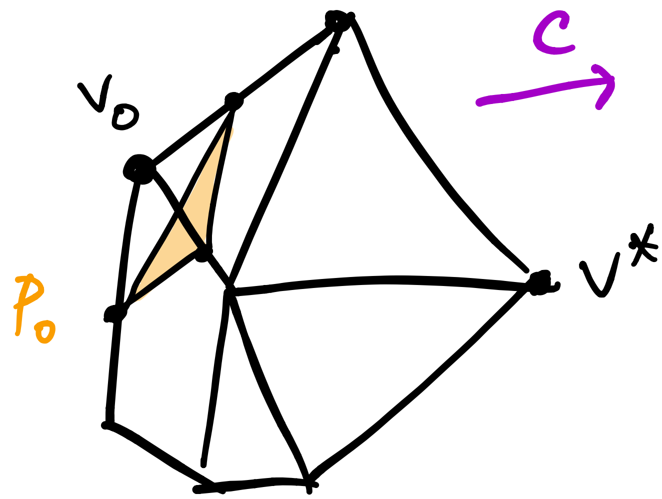
- Enough to show that $\forall$ vertices $v_0 \neq P$, $\exists$ edge to vertex v_1 w/ $\boxed{c^T v_1 > c^T v_0.}$

(by finiteness of # vertices).

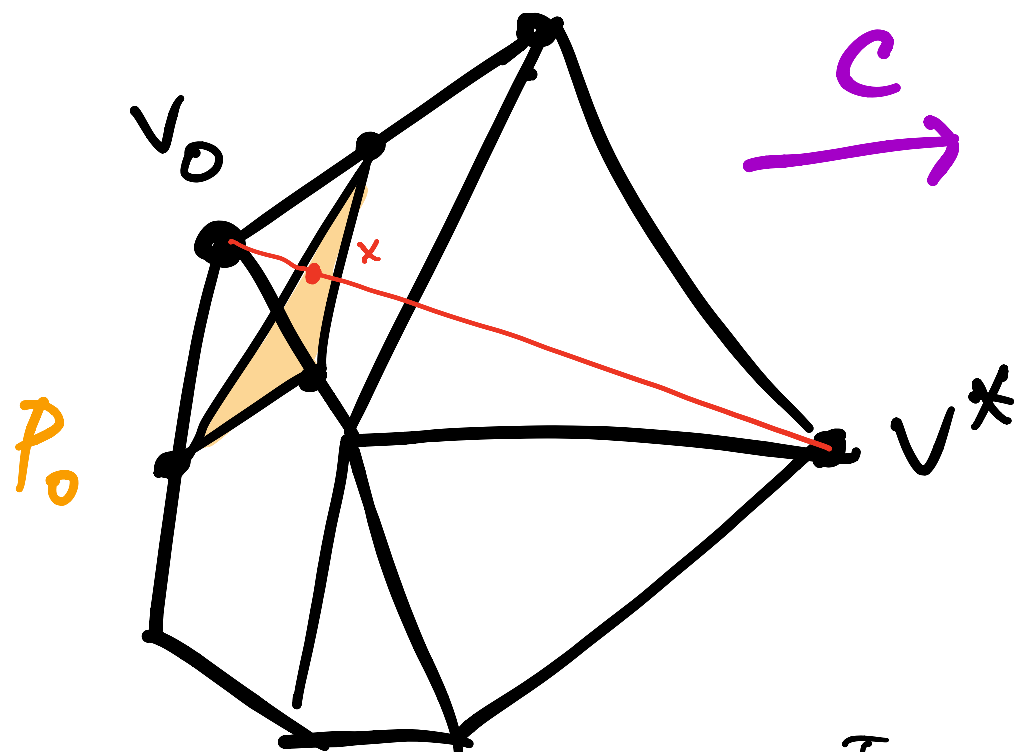




- Let P_0 be polytope from last theorem.



- Let x be intersection of P_0 and segment joining v_0, v^*.



- note that $c^T v_0 < c^T x$.

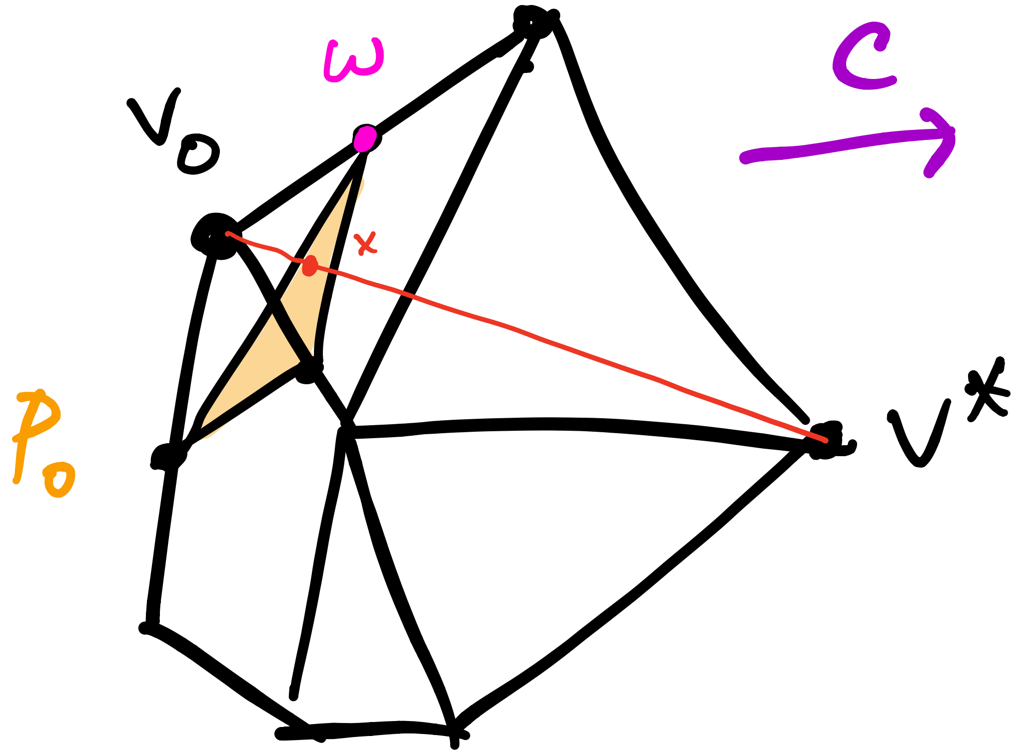
  ($c^T y$ incr. along segment, $v_0 \notin P_0$).

- b/c P_0 polytope,

  $$P_0 = \mathrm{conv}(\text{vertices of } P_0).$$

  $\Rightarrow \exists$ vertex w with

  $$c^T v_0 < c^T x \leq c^T w$$



WHY? Simple but powerful principle:

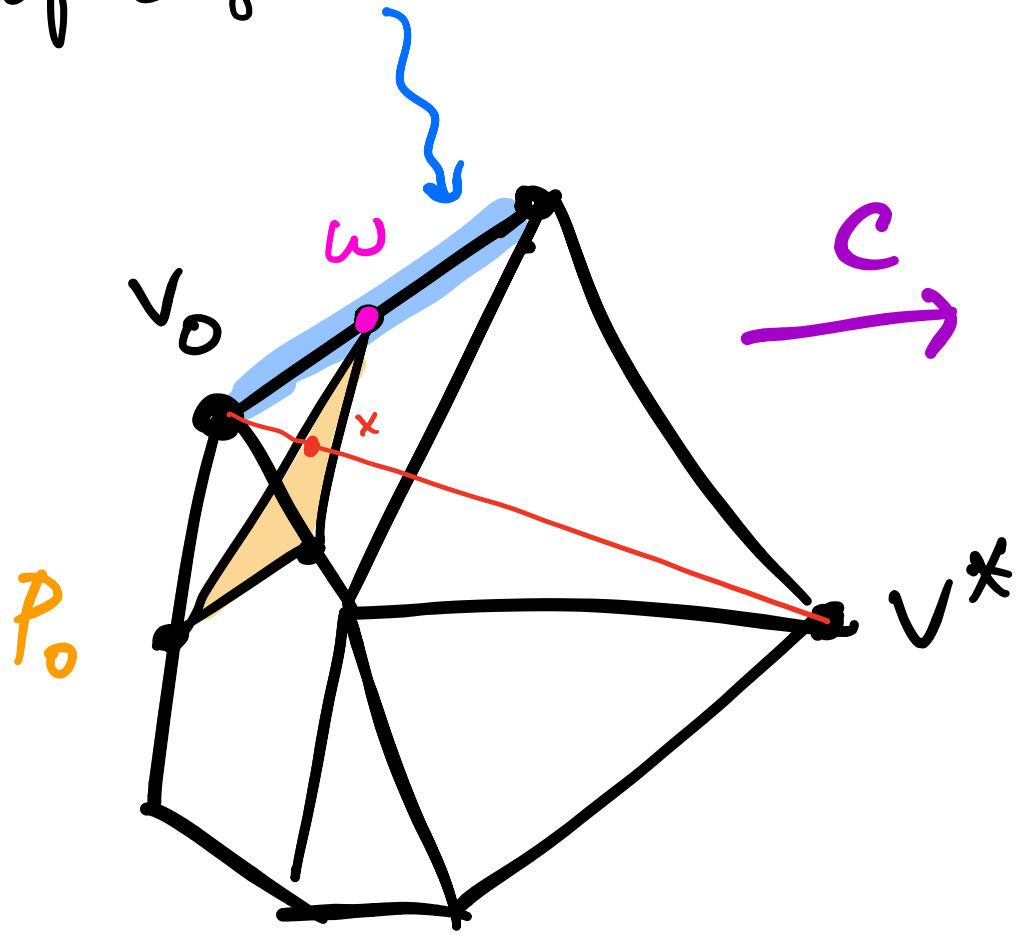
$$x = \sum_{\substack{\omega \\ \text{vertices} \\ \text{of } P_0}} \lambda_\omega \omega, \quad \sum \lambda_\omega = 1$$

$$\Rightarrow c^T x = \sum \lambda_\omega c^T \omega$$ "weighted average"

$$\Rightarrow \text{some } c^T \omega \geq c^T x \text{).}$$

• P... ...iection ω is intersection

- By bijection,
of edge e with P_0.



- e must be bounded
(b/c $c^T y$ increases along e, but objective bounded on P).

- Thus ends at some vertex v_1,

$$c^T v_1 > c^T v_0 \quad \square.$$

Proof of N.V.C:

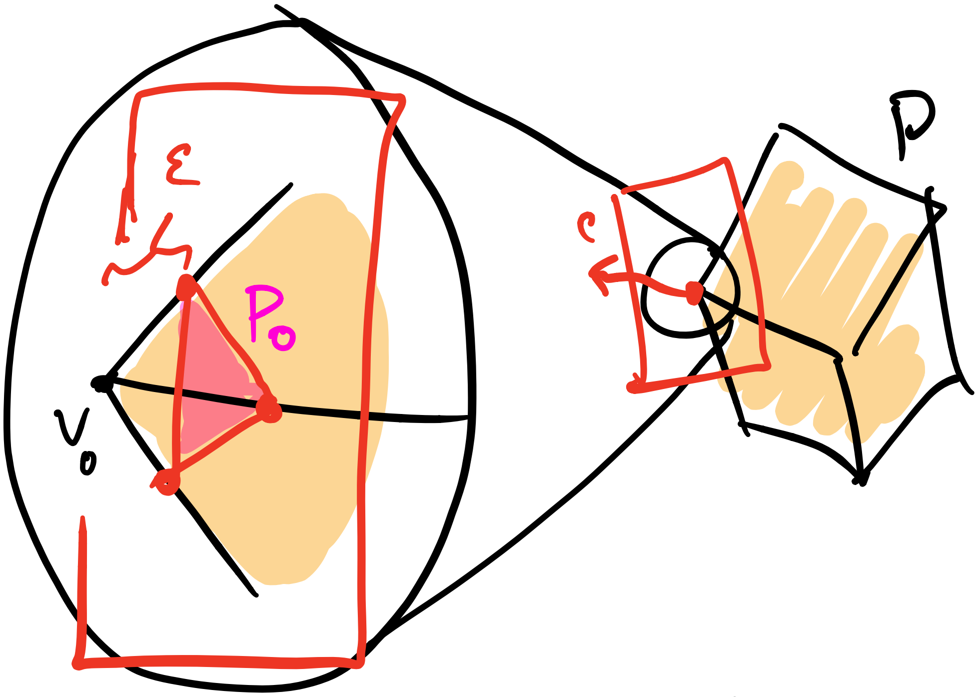
Recall: if vertex v_0 given by

$$c^T x = m,$$

then

$$P_0 = \{x \in P : c^T x = m - \varepsilon\}$$

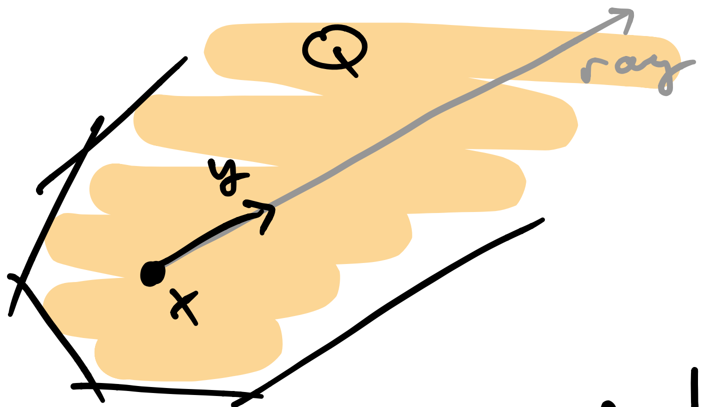
for small ε.



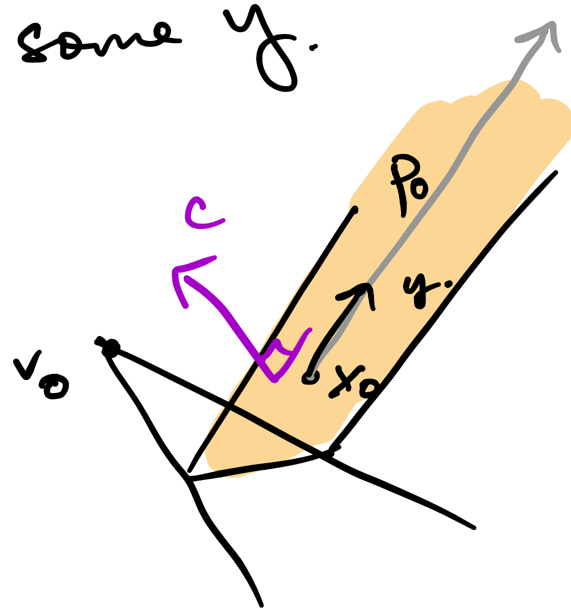
Assume rank $A = n$; else <u>no vertices</u>.

① P_0 bounded.

- <u>Exercise</u>: If Q unbounded polyhedron, $x \in Q$, then Q contains ray from x: $\{x + \alpha y : \alpha \geq 0\}$.

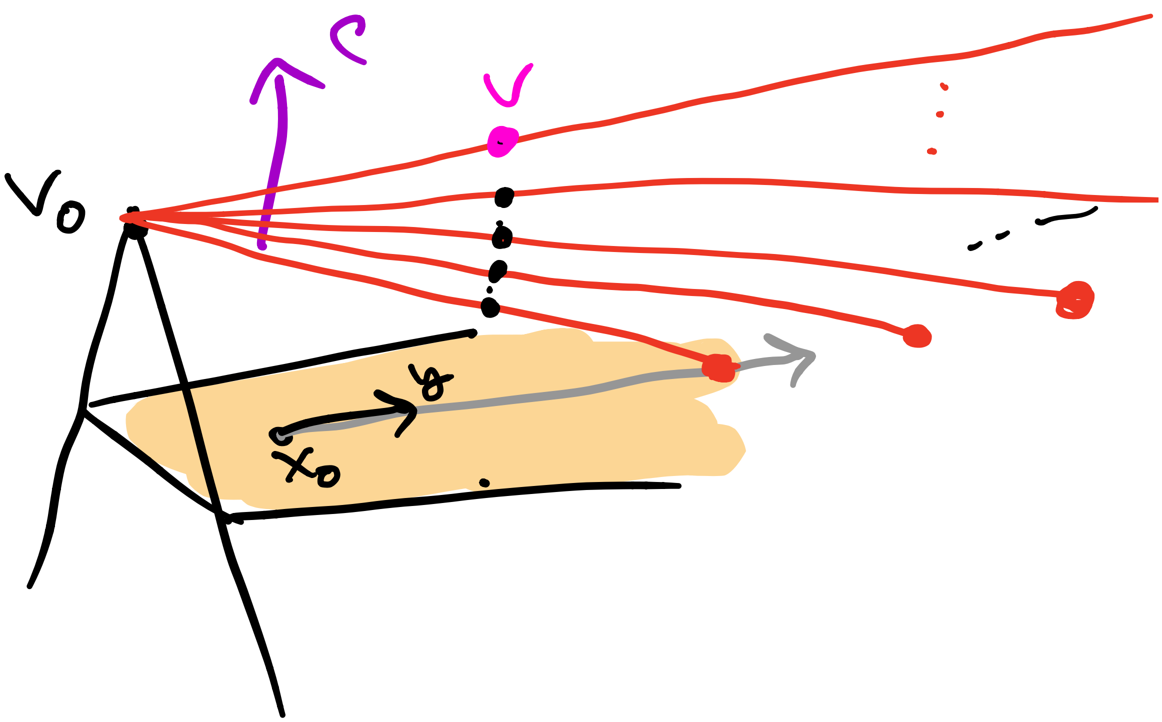


- Suppose P_0 unbounded,
  let $x_0 \in P_0$.
  $\Rightarrow P_0$ contains ray
  $$\{x_0 + \alpha y : \alpha \geq 0\}$$
  for some y.



- as $P_0 \subseteq x_0 + c^\perp$, $y \in c^\perp$.

- use ray to construct another minimizer v contradicting uniqueness:



- By closedness,

$$\{v_0 + \alpha y : \alpha \geq 0\},$$

but $c^T x$ constant along it. ↯

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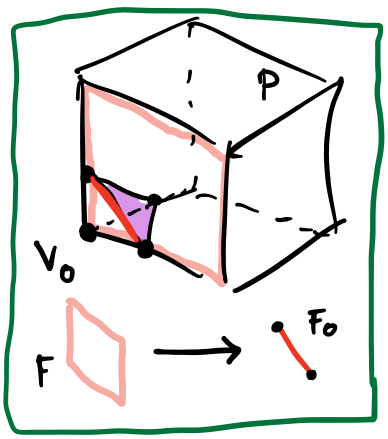
2. The bijection:

$$\text{face } F \ni v_0 \text{ of } P \longrightarrow F_0 := \{x : c^T x = m - \varepsilon\} = F \cap P_0 .$$



a. **onto:** every face F_0 of P_0 can be written this way for some F of P.

- Let F_0 nonempty face of P_0.

$$F_0 = \begin{cases} a_i^T x = b_i & i \in I \\ c^T x = m - \epsilon & \\ a_j^T x \le b_j & j \notin I \end{cases}$$

Let

$$F_0 = \begin{cases} a_i^T x = b_i & i \in I \\ a_j^T x \le b_j & j \notin I \end{cases}$$

(remove middle equality)

- F_0 <u>is</u> a face by faces thm, so just need to show $v_0 \in F_0$.

- Recall that v_0 was only vertex v with

$$c^T v \geqslant m-\epsilon.$$

- But $c^T x$ bounded above on F
  $\Rightarrow$ reaches max $\geqslant m-\epsilon$ at vertex v of F; thus $v=v_0$.

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⑥ Dimensions

(will also imply $\boxed{\text{one-to-one}}$).

- want to show

$$\dim F_0 = \dim F - 1.$$

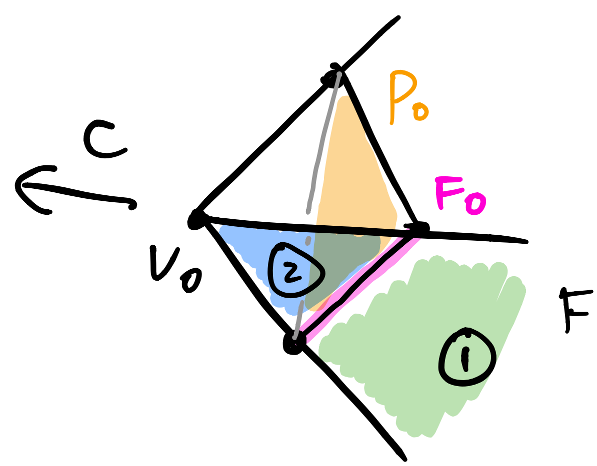
- Enough to show

$\Rightarrow \quad F \subseteq \mathrm{aff}(F_0 \cup \{v_0\})$.

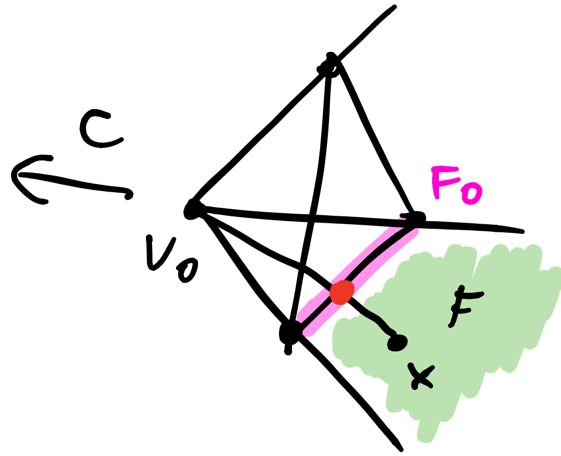
$\Rightarrow \dim F_0 \geq \dim F - 1$.

($\dim F_0 \leq \dim F - 1$ bc $F_0 = F \cap$ plane, $F_0 \neq F$).

Cases: 1 $\boxed{c^T x \leq m - \epsilon}$, 2 $\boxed{c^T x > m - \epsilon.}$



1 If $c^T x \leq m - \epsilon$, segment $x \to v_0$ clearly hits F_0, thus $x \in \mathrm{aff}(F_0 \cup \{v_0\})$.



2 Else, x is in polyhedron

$$F' = F \cap \{x \mid c^T x \geq m - \epsilon\}.$$

- F' is bounded (for same reason as P_0).

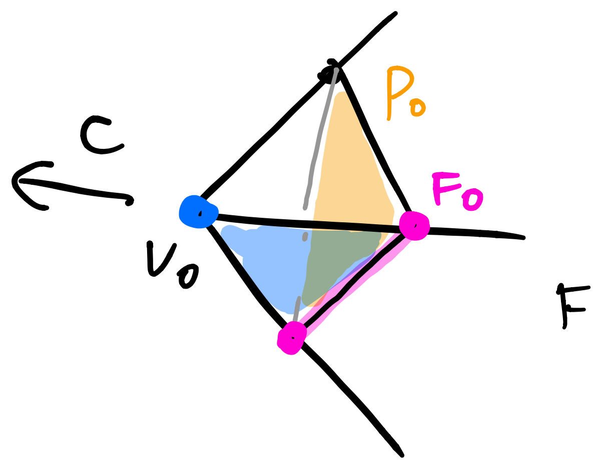
  $\Rightarrow F'$ convex hull of its vertices.

- vertices of F' are all either

  a) on $c^T x = m - \epsilon$ or

b) equal to v_0.

(b/c they are vertices v of F satisfying $c^T v \geq m - t_0$; v_0 only such vertex).



$\Rightarrow F' \subseteq \text{conv}(F_0 \cup \{v_0\})$.

$\subseteq \text{aff}(F_0 \cup \{v_0\})$.